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PLANE WAVES AND HADAMARD STABILITY IN GENERALIZED THIN ELASTIC RODS

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Abstract—In this paper we analyse the propagation of plane waves in generalized thin elastic rods. We obtain relations between Hadamard stability and criteria that wave frequencies be real. Certain design estimates for Hadamard stability in helical and straight rods are also presented.

INTRODUCTION

Ericksen[1] and Shahinpoor[2] have recently given analyses of propagation of plane waves in thin elastic plates and circular cylindrical shells, respectively; they have obtained relations between Hadamard stability and criteria that wave frequencies be real. They have derived the *condition of hyperellipticity* for thin elastic plates[1] and the *restricted condition of hyperellipticity* for thin elastic circular cylindrical shells[2]. In these media plane waves are generally dispersive and generate acoustical activity.

In this paper I intend to do the analogue of [1] and [2] for generalized thin elastic helical rods. I first derive the governing equations of motion from an isochronous Hamilton's principle in N-dimensional Euclidean spaces, employing the kinematical variables proposed by Ericksen and Truesdell[3], Cohen[4], and Antman[5]. I then give an analysis of plane waves in rods whose reference equilibrium configurations are uniform, in the sense of Ericksen[6]. This essentially means that the reference equilibrium configuration is chosen in the form of a rod‡ curved and twisted spirally around a hypothetical fixed circular cylinder. Straight rods are included as special cases. I show that a criterion for reality of frequencies of plane vibrations implies Hadamard stability. I also discuss the phenomena of dispersion and acoustical activity of plane waves in rods. Finally, I present certain design estimates for Hadamard stability in helical and straight rods. I do not mention all works in the area of elastic rods. However, I must mention that Mindlin and Hermann[7], Mindlin and Mcniven[8] and Medick[9] have analysed certain properties of acceleration waves in straight rods. Furthermore, Whitman and Desilva[10] have obtained sufficient criteria for stability and uniqueness of solutions in elastic rods under dissipative forces and couples and Antman[11] has shown that certain instabilities in elastic rods can be interpreted as what is commonly called "necking".

2. GOVERNING EQUATIONS

In the general theory of motion of rods [3–5], a rod is considered as being a smooth curve C_r embedded in a three-dimensional Euclidean space E_3, its position vector from some fixed origin being given by

$$\mathbf{r} = \hat{\mathbf{r}}(X, t), \tag{2.1}$$

where X is the material coordinate of the curve C_r and t is the time. Furthermore, to every particle X of C_r, which is assumed to be the loucs of centroids of cross sections of the rod in some reference equilibrium configuration, there are assigned n deformable vectors

$$\mathbf{d}_i = \hat{\mathbf{d}}_i(X, t),\ (i = 1, 2, \ldots, n). \tag{2.2}$$

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As discussed by Antman[5], the Cosserat model of rods (2.1), (2.2) with $n = 2$ has sufficient kinematical structure to acount for torsion, flexure, axial extensions, transverse extensions, as well as rotory inertia. We assume that $\mathbf{r}$ and $\mathbf{d}_i$ are of class C^2.

The total energy E, associated with C_r is assumed to be of the form

$$E = \int_{C_r} (W + K)\,\mathrm{d}X, \tag{2.3}$$

where W, K are, respectively, the densities of strain and kinetic energies. We assume that the curve C_r is elastic in the sense that the constitutive equations are†

$$W = \hat{W}(r_{,X}; d_i; d_{j,X}; X),\ i,j = 1, 2, \ldots, n, \tag{2.4}$$

$$2K = \rho(X)\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + 2\rho_i(X)\dot{\mathbf{r}}\cdot\dot{\mathbf{d}}_i + \rho_{ij}(X)\dot{\mathbf{d}}_i\cdot\dot{\mathbf{d}}_j \geq 0 \tag{2.5‡}$$

$$K = 0 \leftrightarrow \dot{\mathbf{r}} = \dot{\mathbf{d}}_i = 0, \tag{2.6}$$

where ρ is the mass density, ρ_i are scalars having the dimension of mass density, and $\rho_{ij} = \rho_{ji}$ are the effective inertial mass densities associated with director velocities $\dot{\mathbf{d}}_i$ and $\dot{\mathbf{d}}_j$.

As usual W is assumed to be invariant under static rigid translation and rotation in order to be objective.§ This implies that

$$\hat{W}(R\mathbf{r}_{,X}; R\mathbf{d}_i; R\mathbf{d}_{j,X}; X) = \hat{W}(\mathbf{r}_{,X}; \mathbf{d}_i; \mathbf{d}_{j,X}; X), \tag{2.7}$$

where R is a proper orthogonal transformation in E_3, i.e.

$$R^{-1} = R^T, \det\cdot R = 1. \tag{2.8}$$

Following Ericksen[1], we introduce a $3(n+1)$–dimensional Euclidean space $\mathscr{E}$. Let us take $(n+1)$ vectors $\{\mathbf{r}, \mathbf{d}_i\}$ in E_3 and generate from them a $3(n+1)$–dimensional space in which each vector is an ordered set of vectors in E_3 associated with the ordered set $\mathbf{P} \equiv \{\mathbf{r}, \mathbf{d}_i\}$. Throughout the paper bold-face and script capital letters denote, respectively, vectors and linear transformations in $\mathscr{E}$. For example, if $\mathbf{A} \equiv (\mathbf{a}, \mathbf{b}_i) \epsilon \mathscr{E}$, $\mathscr{K}$ is defined as a linear transformation on $\mathbf{A}$ such that

$$\mathscr{K}\mathbf{A} = (\rho\mathbf{a} + \rho_i\mathbf{b}_i, \rho_j\mathbf{a} + \rho_{ij}\mathbf{b}_i)\epsilon\mathscr{E}. \tag{2.9}$$

In $\mathscr{E}$, the three-dimensional scalar product induces one inner product. If $\mathbf{A}_1 \equiv (\mathbf{a}_1, \mathbf{b}_{1i})$, $\mathbf{A}_2 \equiv (\mathbf{a}_2, \mathbf{b}_{2i})\epsilon\mathscr{E}$, then the above inner product is denoted by

$$\mathbf{A}_1 \circ \mathbf{A}_2 = \mathbf{a}_1\cdot\mathbf{a}_2 + \mathbf{b}_{1i}\cdot\mathbf{b}_{2i}. \tag{2.10}$$

It is clear that $\mathscr{K}$ is symmetric, i.e.

$$\mathbf{A}_1 \circ \mathscr{K}\mathbf{A}_2 = \mathscr{K}\mathbf{A}_1 \circ \mathbf{A}_2, \tag{2.11}$$

and from (2.5) it is positive definite, i.e.

$$2K = \dot{\mathbf{P}}_1 \circ \mathscr{K}\dot{\mathbf{P}}_1 \geq 0,\ \forall \dot{\mathbf{P}}_1 \epsilon \mathscr{E}, \tag{2.12}$$

$$\mathscr{K} = \mathscr{K}^T \geq 0, \tag{2.13}$$

with superscript T denoting the transpose. A linear transformation R in E_3 corresponds to a linear transformation $\mathscr{R}$ in $\mathscr{E}$, e.g.

$$(R\mathbf{r}, R\mathbf{d}_i) = \mathscr{R}(\mathbf{r}, \mathbf{d}_i) = \mathscr{R}\mathbf{P}. \tag{2.14}$$

†Throughout the paper commas denote partial derivatives with respect to material coordinate X and superposed dots denote partial derivatives with respect to time.

‡Summation is applied on indices i and j.

§This also implies invariance under time dependent rigid body motions.

The converse, however, is not generally true, i.e. there exists no linear transformation $\bar{R}$ in E_3 such that $(\bar{R}\mathbf{r}, \bar{R}\mathbf{d}_i) = \mathcal{K}\mathbf{P}$.

In what follows the ordered set $\mathbf{P} \equiv (\mathbf{r}, \mathbf{d}_i)$ plays the role of a set of generalized coordinates associated with rods. From (2.7)–(2.14) we have

$$W = \hat{W}(\mathbf{P}, \mathbf{P}_{,X}, X) = \hat{W}(\mathcal{R}\mathbf{P}, \mathcal{R}\mathbf{P}_{,X}, X), \tag{2.15}$$

$$\mathbf{P} = (\mathbf{r}, \mathbf{d}_i), \tag{2.16}$$

where $\mathbf{P}$ plays the role of generalized coordinate in $\mathscr{C}$. Let $[t_0, t_1]$ denote a closed time interval with t_0 and t_1 fixed, and let $[X_0, X_1]$ denote a fixed material coordinate interval. We now employ an isochronous Hamilton's action principle for the motion of the rod, i.e. we require that

$$\int_{t_0}^{t_1}\left[\left(\int_{X_0}^{X_1} \delta L^* \,\mathrm{d}X\right) + \int_{X_0}^{X_1} (\mathbf{F}^\circ \delta\mathbf{P}) \,\mathrm{d}X\right] \mathrm{d}t + \int_{t_0}^{t_1} [\mathbf{T}^\circ \delta\mathbf{P}]_{X_0}^{X_1} \,\mathrm{d}t = 0, \tag{2.17}$$

$$\delta\mathbf{P}(X, t_0) = \delta\mathbf{P}(X, t_1) = 0, \tag{2.18}$$

for all arbitrary intervals $[t_0, t_1]$, and all arbitrary variations $\delta\mathbf{P}$. In (2.17), L^* is the Lagrangian, given by

$$L^* = \tfrac{1}{2}\dot{\mathbf{P}}^\circ \mathcal{K}\dot{\mathbf{P}} - \hat{W}(P, P_{,X}, X), \tag{2.19}$$

$\mathbf{F}$ is the generalized body force per unit material length, and $\mathbf{T}$ is the generalized boundary traction and couple. Performing the variation in (2.17) and employing the divergence theorem, assuming C_r and $\mathbf{P}$ to be sufficiently smooth, we obtain the following governing equations of motion and boundary conditions, respectively:

$$\left[\frac{\partial W}{\partial \mathbf{P}_{,X}}\right]_{,X} - \frac{\partial W}{\partial \mathbf{P}} + \mathbf{F} = \mathcal{K}\ddot{\mathbf{P}} \tag{2.20}$$

$$\mathbf{T}(X_0, t) = \left[\frac{\partial W}{\partial \mathbf{P}_{,X}}\right]_{X_0}, \tag{2.21}$$

$$\mathbf{T}(X_1, t) = \left[\frac{\partial W}{\partial \mathbf{P}_{,X}}\right]_{X_1} \tag{2.22}$$

With the notations

$$\mathbf{T} \equiv \frac{\partial W}{\partial \mathbf{P}_{,X}}, \mathbf{M} \equiv -\frac{\partial W}{\partial \mathbf{P}}, \tag{2.23}$$

the condition for static equilibrium in the reference configuration is found from (2.20) to be

$$\bar{\mathbf{T}}_{,X} + \bar{\mathbf{M}} + \bar{\mathbf{F}} = 0, \tag{2.24}$$

where the bar denotes evaluation in the reference equilibrium configuration.

3. DESCRIPTION OF THE REFERENCE EQUILIBRIUM CONFIGURATION

We consider a reference equilibrium configuration for the rod†, uniform in the sense of Ericksen[6], in the form of a cylindrical helix;

$$\bar{\mathbf{r}} \equiv [a \cos bX, a \sin bX, cX], \tag{3.1}$$

$$\bar{\mathbf{d}}_i \equiv [b_i \cos(bX + C_i), b_i \sin(bX + C_i), e_i], \tag{3.2}$$

where a, b, c, b_i, C_i, and e_i are fixed constants for each given configuration. Equation (3.2) implies

†An oriented curve.

that each $\mathbf{d}_i$, (i fixed), makes fixed angles with the tangent, the binormal and the principal normal to the line, while retaining a fixed magnitude. From (3.1) and (3.2) it is clear that $a = 0$, corresponds to a straight rod which is generally twisted, while $c = 0$, corresponds to a circular rod which is generally twisted. The value of $\bar{\mathbf{F}}$ necessary to maintain a uniform reference equilibrium configuration (3.1), (3.2) should be calculated from (2.24).

We now make the following change of variables

$$\mathbf{r} = \bar{\mathbf{r}} + \hat{R}\mathbf{p}, \mathbf{d}_i = \hat{R}\mathbf{q}_i, \tag{3.3}$$

where

$$R = \begin{bmatrix} \cos bX & -\sin bX & 0 \\ \sin bX & \cos bX & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.4}$$

Note that $\hat{R} \overset{T}{=} \hat{R}^{-1}$, $\det \hat{R} = 1$. Further, it is easily shown that

$$\hat{R}^T\bar{\mathbf{r}}_{,X} = \mathbf{a}_X \equiv (o, ab, c), \tag{3.5}$$

$$\hat{R}_{,X} = \hat{R}S_X = S_X\hat{R}, \tag{3.6}$$

$$S_X = -S_X{}^T = b\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.7}$$

$$\bar{\mathbf{q}}_i = \text{Const.}, \bar{\mathbf{p}} = 0. \tag{3.8}$$

In abbreviated $3(n+1)$–dimensional notation,

$$\bar{P} = (\bar{r}, o) + \hat{\mathcal{R}}\mathbf{Q}, \tag{3.9}$$

$$\mathbf{Q} = (\mathbf{p}, \mathbf{q}_i), \tag{3.10}$$

$$\mathbf{P}_{,X} = \hat{\mathcal{R}}\mathbf{T}_X, \tag{3.11}$$

$$\mathbf{T}_X = \mathbf{A}_X + \mathcal{S}_X\mathbf{Q} + \mathbf{Q}_{,X}, \tag{3.12}$$

$$\hat{\mathcal{R}}_{,X} = \mathcal{S}_X\hat{\mathcal{R}} = \hat{\mathcal{R}}\mathcal{S}_X, \tag{3.13}$$

$$\mathbf{A}_X = (\mathbf{a}_X, \mathbf{o}), \tag{3.14}$$

and $\mathcal{S}_X$ is *a* $n \times n$ diagonal matrix whose diagonal elements are 3×3 matrices such that

$$\mathcal{S}_X = -\mathcal{S}_X{}^T = S_X\delta_{ij}, \tag{3.15}$$

where δ_{ij} is *a* $n \times n$ Kroncker delta.

From (3.1–3.15) we have for homogeneous materials only,

$$\hat{W}(\mathbf{P}, \mathbf{P}_{,X}) \equiv \hat{W}(\hat{\mathcal{R}}^T\mathbf{P}, \hat{\mathcal{R}}^T\mathbf{P}_{,X}) = \hat{W}(\mathbf{Q}, \mathbf{T}_X), \tag{3.16}$$

$$\frac{\partial \hat{W}}{\partial \mathbf{P}} \equiv \hat{\mathcal{R}}\frac{\partial \hat{W}}{\partial \mathbf{Q}}, \frac{\partial \hat{W}}{\partial \mathbf{P}_{,X}} \equiv \hat{\mathcal{R}}\frac{\partial \hat{W}}{\partial \mathbf{T}_X}. \tag{3.17}$$

We now introduce a new function $\hat{B}$ such that

$$\hat{B}(\mathbf{Q}, \mathbf{Q}_{,X}) \equiv \hat{W}(\mathbf{Q}, \mathbf{T}_X). \tag{3.18}$$

Of course $\hat{B}$ will be different for different choices of $\bar{\mathbf{r}}$, $\hat{\mathbf{R}}$, insofar as they give different values for the constants $\mathbf{A}_X$ and $\mathcal{S}_X$. It follows that

$$\frac{\partial \hat{B}}{\partial \mathbf{Q}} = \frac{\partial \hat{W}}{\partial \mathbf{Q}} - \mathcal{S}_X \frac{\partial \hat{W}}{\partial \mathbf{T}_X}, \tag{3.19}$$

$$\frac{\partial \hat{B}}{\partial \mathbf{Q}_{,X}} = \frac{\partial \hat{W}}{\partial \mathbf{T}_X}. \tag{3.20}$$

The governing equation (2.20) now reduces to

$$\left[\frac{\partial \hat{B}}{\partial \mathbf{Q}_{,X}}\right]_{,X} - \left[\frac{\partial \hat{B}}{\partial \mathbf{Q}}\right] + \mathbf{G} = \mathcal{K}\ddot{\mathbf{Q}}, \tag{3.21}$$

$$\mathbf{G} \equiv \hat{\mathcal{R}}^{-1}\mathbf{F}. \tag{3.22}$$

4. LINEARIZATION AND AN ANALYSIS OF PLANE WAVES IN RODS

It is clear that for the reference state $\bar{C}_r$, $\mathbf{Q} = \bar{\mathbf{Q}} =$ constant. To maintain such a ground state in equilibrium one should calculate the necessary constant value of $\mathbf{G}$. For example if $\mathbf{F} = \hat{\mathcal{R}}\bar{\mathbf{F}}$ with $\bar{\mathbf{F}}$ denoting a specified constant generator of the dead load in C_r, then $\mathbf{G} = \bar{\mathbf{G}} = \bar{\mathbf{F}}$. We proceed to linearize about $\bar{\mathbf{Q}}$ by carrying out the well-defined mathematical process of linearization consisting of taking the first variation or Fréchet derivative of the governing equations. To do this we put $\mathbf{Q} = \bar{\mathbf{Q}} + \epsilon\mathbf{V}$ into the governing equations, differentiate with respect to ϵ and let $\epsilon \to 0$. The resulting equations are:

$$\mathcal{L}^{XX}V_{,XX} + \mathcal{L}^{X}V_{,X} + \mathcal{L}\mathbf{V} = \mathcal{K}\ddot{\mathbf{V}}, \tag{4.1}$$

where

$$\mathcal{L}^{XX} = \overline{\frac{\partial^2 \hat{B}}{\partial \mathbf{Q}_{,X}\,\partial \mathbf{Q}_{,X}}} = \mathcal{L}^{XXT}, \tag{4.2}$$

$$\mathcal{M}^{X} = \overline{\frac{\partial^2 \hat{B}}{\partial \mathbf{Q}\,\partial \mathbf{Q}_{,X}}}, \tag{4.3}$$

$$\mathcal{L} = \overline{\frac{-\partial^2 \hat{B}}{\partial \mathbf{Q}\,\partial \mathbf{Q}}} = \mathcal{L}^{T} \tag{4.4}$$

$$\mathcal{L}^{X} = \mathcal{M}^{XT} - \mathcal{M}^{X} = -\mathcal{L}^{XT}, \tag{4.5}$$

where $\mathcal{L}^{XX}$, $\mathcal{M}^{X}$, $\mathcal{L}^{X}$, and $\mathcal{L}$ are all $3(n+1)\times 3(n+1)$ constant matrices and bar denotes evaluation at $\mathbf{Q} = \bar{\mathbf{Q}}$.

In order to analyse plane waves we consider solutions of the form

$$\mathbf{V} = \mathbf{A}\,e^{i(kX-\omega t)}, \tag{4.6}$$

where $\mathbf{A}$ and ω are generally complex constants while k is assumed to be real. The substitution of (4.6) and (4.1) reduces the latter to

$$(\mathcal{H} - \omega^2\mathcal{K})\mathbf{A} = 0, \tag{4.7}$$

where

$$\mathcal{H} = \mathcal{L}^{XX}k^2 - i\mathcal{L}^{X}k - \mathcal{L} = \mathcal{H}(k) \tag{4.8}$$

It is clear from (4.2), (4.4), (4.5) and (4.8) that

$$\mathcal{H}^*(k) = \mathcal{H}^T(k) = \mathcal{H}(-k), \tag{4.9}$$

and therefore $\mathcal{H}$ is Hermitian and generally complex. It follows that the roots $\omega^2(k)$ obtained from

$$\det.(\mathscr{H} - \omega^2\mathscr{K}) = 0, \tag{4.10}$$

are all real which means ω is either real or pure imaginary. Because of the properties of $\mathscr{H}$ and $\mathscr{K}$ for all complex $3(n+1)$–vectors $\mathbf{B}\epsilon\mathscr{C}$, the following quadratic forms are real

$$\Phi = \mathbf{B}^{*\circ}\mathscr{H}\mathbf{B}, \Psi = \mathbf{B}^{*\circ}\mathscr{K}\mathbf{B} \geq 0. \tag{4.11}$$

The eigenvalue problem (4.7) defines the following extremum problem

$$\omega^2 = \text{ext.}\,[\Phi/\Psi], (k \text{ fixed}), \tag{4.12}$$

where $3(n+1)$ extremum values are denoted by ω_N^2, $(N = 1, 2, \ldots, 3(n+1))$. Note that $\omega_N^2 \geq 0$ if and only if $\Phi \geq 0 \forall \mathbf{B}$ or equivalently

$$\mathbf{B}^{*\circ}\mathscr{H}\mathbf{B} \geq 0, \forall \mathbf{B}\epsilon\mathscr{C} \text{ and real } k. \tag{4.13}$$

Since k is real, (4.13) can be rewritten as

$$i\mathbf{B}k^\circ\mathscr{L}^{XX}(i\mathbf{B}k)^* + \mathbf{B}^\circ\mathscr{L}^X(i\mathbf{B}k)^* - \mathbf{B}^\circ\mathscr{L}\mathbf{B}^* \geq 0. \tag{4.14}$$

Condition (4.14), which is a criterion for the reality of frequencies of plane waves in rods is called *condition of hyperellipticity*, in rods. It is straight forward to show that, with ω real, one solution (4.6) generates three others so that adding the four gives a real solution of the form

$$\mathbf{V} = (\mathbf{D}\cos kX + \mathbf{E}\sin kX)\cos\omega t, \tag{4.15}$$

$$\mathbf{D} = \mathbf{A} + \mathbf{A}^*, \mathbf{E} = i(\mathbf{A} - \mathbf{A}^*). \tag{4.16}$$

From (4.15) it is seen that, when (4.14) is satisfied, then for each k there exist $3(n+1)$ Fourier "plane wave" components each travelling at its own speed and frequency. The above observation, thus, gives rise to dispersion and acoustical activity. If $\mathbf{D} = \alpha\mathbf{E}$, with α an arbitrary real constant, then it follows from (4.16) that $\mathbf{A}$ can be taken to be real. This is an exceptional case unless the ground state is such that

$$\mathscr{L}^X = 0 \Rightarrow \overline{\frac{\partial^2\hat{B}}{\partial\mathbf{Q}\,\partial\mathbf{Q}_{,X}}} = \overline{\frac{\partial^2\hat{B}}{\partial\mathbf{Q}_{,X}\,\partial\mathbf{Q}}}, \tag{4.17}$$

in which case there is no acoustical activity, though the plane waves are still dispersive in general.

5. STATIC STABILITY ANALYSIS

In order to study the static stability of the reference equilibrium configuration, we should consider the change in energy δE of an isolated system, defined as

$$\delta E = \delta E_0 + \delta E_1 + \delta E_2, \tag{5.1}$$

where

$$\delta E_0 = \int_{X_0}^{X_1} [W(\mathbf{P}, \mathbf{P}_{,X}) - \bar{W}(\bar{\mathbf{P}}, \bar{\mathbf{P}}_{,X})]\,\mathrm{d}X \tag{5.2}$$

$$= \int_{X_0}^{X_1} [\hat{B}(\mathbf{Q}, \mathbf{Q}_{,X}) - \bar{\hat{B}}(\mathbf{Q}, 0)]\,\mathrm{d}X, \tag{5.3}$$

$$\delta E_1 = -\int_{X_0}^{X_1} \mathbf{F}^\circ(\mathbf{P} - \bar{\mathbf{P}})\,\mathrm{d}X = -\int_{X_0}^{X_1} \mathbf{G}^\circ\mathbf{V}\,\mathrm{d}X, \tag{5.4}$$

and δE_2 is the change in energy due to end loads. From (5.1–5.4) we find that

$$\delta E = \int_{X_0}^{X_1} \chi(\mathbf{V}, \mathbf{V}_{,X})\,\mathrm{d}X + [\bar{\bar{\mathbf{T}}}\circ\mathbf{V}]_{X_0}^{X_1} + \delta E_2, \tag{5.5}$$

where

$$\chi(\mathbf{V}, \mathbf{V}_{,X}) = \hat{B}(\mathbf{Q}, \mathbf{Q}_{,X}) - \hat{B}(\bar{\mathbf{Q}}, 0) - \bar{\bar{\mathbf{T}}}\circ\mathbf{V}_{,X} + \bar{\bar{\mathbf{G}}}\circ\mathbf{V} \tag{5.6}$$

$$\bar{\bar{\mathbf{T}}} \equiv \overline{\frac{\partial \hat{B}}{\partial \mathbf{Q}_{,X}}}, \quad \bar{\bar{\mathbf{G}}} \equiv -\overline{\frac{\partial \hat{B}}{\partial \mathbf{Q}}}. \tag{5.7}$$

From the remarks in section 4, if we take δE to be the first variation of E, then

$$2\chi = \mathbf{V}_{,X}\circ\mathscr{L}^{XX}\mathbf{V}_{,X} + 2\mathbf{V}\circ\mathscr{M}^{X}\mathbf{V}_{,X} - \mathbf{V}\circ\mathscr{L}\mathbf{V}. \tag{5.8}$$

We employ the traditional static stability criterion:

$$\delta E \geq 0, \tag{5.9}$$

for all $\mathbf{V}$ that are continuous on the closure of C_r, square integrable along with $\mathbf{V}_{,X}$, and consistent with any kinematical constraints imposed by the edge loadings. Knops and Wilkes[12] have discussed why (5.9), with δE interpreted as the first variation in E, may not be a sound criterion for stability. Let us consider a common loading device, one most likely to promote stability, in the form of hyperclamps, i.e. completely glued or "welded" boundary cross sections, by setting

$$\mathbf{V}(X_0) = \mathbf{V}(X_1) = 0, \delta E_2 = 0. \tag{5.10}$$

Condition (5.7) reduces the criterion (5.9) to

$$\int_{X_0}^{X_1} \chi(\mathbf{V}, \mathbf{V}_{,X})\,\mathrm{d}X \geq 0, \mathbf{V}(X_0) = \mathbf{V}(X_1) = 0. \tag{5.11}$$

We refer to (5.11) as the Hadamard stability criterion for generalized elastic rods. We follow the approach of van Hove[13] and extend the domain of $\mathbf{V}$ to the entire line in the reference equilibrium configuration, setting

$$\mathbf{V} = 0 \text{ outside } [X_0, X_1]. \tag{5.12}$$

Employing the Fourier transforms

$$\hat{\mathbf{V}}(k) = \sqrt{\frac{1}{2\pi}} \int_{X_0}^{X_1} \mathbf{V}\, \mathrm{e}^{ikX}\,\mathrm{d}X, \tag{5.13}$$

$$-ik\hat{\mathbf{V}}(k) = \sqrt{\frac{1}{2\pi}} \int_{X_0}^{X_1} \mathbf{V}_{,X}\, \mathrm{e}^{ikX}\,\mathrm{d}X, \tag{5.14}$$

the inverse transforms

$$\mathbf{V} = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{V}}(k)\, \mathrm{e}^{-ikX}\,\mathrm{d}k, \tag{5.15}$$

$$\mathbf{V}_{,X} = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} (-ik)\, \hat{\mathbf{V}}(k)\, \mathrm{e}^{-ikX}\,\mathrm{d}k, \tag{5.16}$$

and Parseval's theorem, with $\mathbf{V}$ real, we obtain

$$\int_{-\infty}^{\infty} \hat{\mathbf{V}} \otimes \hat{\mathbf{V}}^* \, dk = \int_{X_0}^{X_1} \mathbf{V} \otimes \mathbf{V} \, dX \tag{5.17}$$

$$\int_{-\infty}^{\infty} \hat{\mathbf{V}} \otimes (-ik\hat{\mathbf{V}})^* \, dk = \int_{X_0}^{X_1} \mathbf{V} \otimes \mathbf{V}_{,X} \, dX = -\int_{X_0}^{X_1} \mathbf{V}_{,X} \otimes \mathbf{V} \, dX, \tag{5.18}$$

$$\int_{-\infty}^{\infty} (i\hat{\mathbf{V}}k) \otimes (i\mathbf{V}k)^* \, dk = \int_{X_0}^{X_1} \mathbf{V}_{,X} \otimes \mathbf{V}_{,X} \, dX, \tag{5.19}$$

where $\otimes$ denotes the tensor product. Employing (5.17–5.19) in (5.11) we obtain

$$2\int_{X_0}^{X_1} \chi(\mathbf{V}, \mathbf{V}_{,X}) \, dX = \int_{-\infty}^{\infty} [(i\hat{\mathbf{V}}k)^\circ \mathscr{L}^{XX} (i\hat{\mathbf{V}}k)^* + \mathbf{V}^\circ \mathscr{L}^X (i\hat{\mathbf{V}}k)^* - \hat{\mathbf{V}}^\circ \mathscr{L} \hat{\mathbf{V}}] \, dk \geq 0, \tag{5.20}$$

which should hold for all real k,s. We can now state the following theorems:

Theorem 1. The condition of hyperellipticity (4.14) necessary that the frequencies of plane waves be real is a sufficient condition for Hadamard stability of any elastic rod whose reference equilibrium configuration is uniform.

The proof easily follows by comparing (4.14) and (5.20).

Theorem 2. The condition of hyperellipticity (4.14) necessary for the reality of frequencies of plane waves implies the strong-ellipticity condition

$$\mathbf{C}^\circ \mathscr{L}^{XX} \mathbf{C}^* \geq 0, \ \forall \mathbf{C}. \tag{5.21}$$

The proof follows by dividing (4.14) by a positive number k^2 and then letting $k \to \infty$.

Note that (5.21) does not imply (4.14) because the dependence of χ on $\mathbf{V}$ is not restricted.

We now show that (5.21) is necessary for Hadamard stability.† We consider (5.11) and note that $\mathbf{V}$ can be expanded in a Fourier series, i.e.

$$\mathbf{V} = \sum_{n=1}^{\infty} \mathbf{A}_n \sin n\pi\xi, \ \xi = \frac{X - X_0}{X_1 - X_0}, \ X_1 - X_0 = \hat{l}, \tag{5.22}$$

Substituting (5.22) in (5.11) and performing the integration we obtain

$$\sum_{n=1}^{\infty} \left\{ \mathbf{A}_n^\circ \mathscr{L}^{XX} \mathbf{A}_n \frac{n^2\pi^2}{\hat{l}^2} - \left[\sum_{m=1}^{\infty} \mathbf{A}_n^\circ \mathscr{L}^X g_{nm} \mathbf{A}_m \hat{l}^{-1} \right] - \mathbf{A}_n^\circ \mathscr{L} \mathbf{A}_n \right\} \geq 0, \tag{5.23}$$

where

$$g_{nm} = \begin{cases} 0, & n \pm m \text{ even} \\ \dfrac{4nm}{n^2 - m^2}, & n + m \text{ odd}, \ n \neq m. \end{cases} \tag{5.24}$$

Condition (5.23) must hold for all real vectors $\mathbf{A}_n$ and all lengths $\hat{l}$. It follows from (5.23) that

Theorem 3. The strong-ellipticity condition (5.21) is necessary for Hadamard stability.

Proof. In (5.23) let all but one $\mathbf{A}_n = 0$, and take n large and the proof follows.

4. STABILITY DESIGN CRITERIA

As noted before the strong ellipticity condition is necessary for Hadamard stability in rods. On the other hand, I showed that the hyperellipticity condition is sufficient for Hadamard stability. I shall now seek a condition intermediate between the strong ellipticity condition but sufficient to ensure Hadamard stability.

First we note that

$$2\int_{X_0}^{X_1} \chi(\mathbf{V}, \mathbf{V}_{,X}) \, dX \geq \int_{-\infty}^{\infty} [\lambda_1 k^2 (\hat{\mathbf{V}}^\circ \hat{\mathbf{V}}^*) + \hat{\mathbf{V}}^\circ \mathscr{L}^X (ik\hat{\mathbf{V}})^* - \hat{\mathbf{V}}^\circ \mathscr{L} \hat{\mathbf{V}}] \, dk, \tag{6.1}$$

†This is a known result (see Graves[14]). Here, however, we present a simpler approach.

where λ_1 is the smallest eigenvalue of $\mathscr{L}^{XX}$ such that

$$\mathbf{C}^{\circ}\mathscr{L}^{XX}k^2\mathbf{C}^* \geq \lambda_1 k^2(\mathbf{C}^{\circ}\mathbf{C}^*). \tag{6.2}$$

From (5.17–5.19) and (6.1) we obtain

$$2\int_{X_0}^{X_1} \chi \, \mathrm{d}X \geq \int_{X_0}^{X_1} [\lambda_1 \mathbf{V}^{\circ}_{,X}\mathbf{V}_{,X} - \mathbf{V}^{\circ}\mathscr{L}^{X}\mathbf{V}_{,X} - \mathbf{V}^{\circ}\mathscr{L}\mathbf{V}] \, \mathrm{d}X. \tag{6.3}$$

By the Schwarz inequality

$$\left|\int_{X_0}^{X_1} (\mathbf{V}^{\circ}\mathscr{L}^{X}\mathbf{V}_{,X}) \, \mathrm{d}X\right| \leq \left(\int_{X_0}^{X_1} \mathbf{V}^{\circ}\mathbf{V} \, \mathrm{d}X\right)^{1/2}\left(\int_{X_0}^{X_1} (\mathscr{L}^{X}\mathbf{V}^{\circ}\mathscr{L}^{X}\mathbf{V}_{,X}) \, \mathrm{d}X\right)^{1/2}. \tag{6.4}$$

Furthermore,

$$\mathscr{L}^{X}\mathbf{V}^{\circ}_{,X}\mathscr{L}^{X}\mathbf{V}_{,X} \leq \lambda_2(\mathbf{V}^{\circ}_{,X}\mathbf{V}_{,X}), \ \lambda_2 \geq 0, \tag{6.5}$$

$$\mathbf{V}^{\circ}\mathscr{L}\mathbf{V} \leq \lambda(\mathbf{V}^{\circ}\mathbf{V}), \ \lambda = \max(o, \lambda_3), \tag{6.6}$$

where λ_2, λ_3 are, respectively, the largest eigenvalues of the symmetric tensors $\mathscr{L}^{XT}\mathscr{L}^{X}$ and $\mathscr{L}$.

Let us now employ the Poincare's inequality

$$\int_{X_0}^{X_1} (\mathbf{V}^{\circ}\mathbf{V}) \, \mathrm{d}X \leq \hat{p}_1(L) \int_{X_0}^{X_1} (\mathbf{V}^{\circ}_{,X}\mathbf{V}_{,X}) \, \mathrm{d}X, \tag{6.7}$$

$$\mathbf{V}(X_0) = \mathbf{V}(X_1) = 0, \hat{p}_1(L) > 0. \tag{6.8}$$

Employing (5.22) it can be easily shown that the best constant $\hat{p}_1(L)$ is

$$\hat{p}_1(L) = \pi^{-2}\hat{l}^2. \tag{6.9}$$

From (6.3–6.9) we obtain

$$2\int_{X_0}^{X_1} \chi \, \mathrm{d}X \geq \int_{X_0}^{X_1} [\lambda_1 - \lambda_2^{1/2}\pi^{-1}\hat{l} - \lambda\pi^{-2}\hat{l}^{-2}](\mathbf{V}^{\circ}_{,X}\mathbf{V}_{,X}) \, \mathrm{d}X \tag{6.10}$$

Therefore:

Theorem 4. A sufficient condition for the Hadamard stability of any elastic rod possessing a uniform reference equilibrium configuration given by (3.1), (3.2), is

$$(\lambda_1 - \pi^{-1}\hat{l}\lambda_2^{1/2} - \pi^{-2}\hat{l}^2\lambda) \geq 0, \ \lambda = \max(o, \lambda_3), \tag{6.11}$$

where λ_1, λ_2, λ_3, are material constants given by (6.2), (6.5), and (6.6), respectively.

Theorem 5. For all sufficiently short helical rods whole reference equilibrium configuration is given by (3.1), (3.2), the strong form of the condition of strong-ellipticity $[\lambda_1 > 0]$ is sufficient for Hadamard stability.

Proof. Clearly, $\lambda_1 > 0 \Rightarrow$ (6.11) for $\hat{l}$ sufficiently small and the proof follows.

Theorem 6. For the reference equilibrium configuration (3.1), (3.2), strong-ellipticity implies Hadamard stability for all $\hat{l} < \infty$, provided $\mathscr{L}^{X} = 0$, $-\mathscr{L} > 0$.

Proof. Under the above conditions it is obvious that $\lambda_2 = \lambda = 0$ and (6.11) reduces to

$$\lambda_1 \geq 0. \tag{6.12}$$

It appears from the present analysis that the hyperellipticity condition (4.14) is more appropriate for long helical and straight rods. For very short rods the strong ellipticity condition should suffice as a check for stability. For moderate lengths the designer should consult the intermediate criterion (6.11) for Hadamard stability.

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